Interaction of a Rough Subliming Surface and a Laminar Boundary Layer

Paul A. Libby*
AVCO Systems Division, Wilmington, Mass.

The laminar boundary layer at a two-dimensional stagnation point over a subliming surface with an internal structure resulting in roughness is analyzed. A simplifying assumption regarding the rate of recession on the roughness elements leads to pyramidal shape of these elements. A roughness parameter Γ involving inter alia a Reynolds number based on the number of roughness elements per unit area is shown to dominate the boundary-layer behavior; when $\Gamma \sim \infty$, the height of the roughness elements is negligible, but their surface density is large so that the boundary layer is altered; when $\Gamma \ll 1$ the roughness elements extend to the edge of the boundary layer and determine the boundary thickness.

Nomenclature

a	= acceleration parameter, Eqs. (10) and (11)
c_p	= specific heat at constant pressure
c_s	= specific heat of the solid
D	= drag per unit wetted area per unit volume
$\mathfrak{D}$	= diffusion coefficient
f	= modified stream function, Eqs. (11) and (12)
h	= enthalpy of the gas
h_c	= nondimensional heat-transfer coefficient
h_v	= enthalpy of the vapor at a vaporization temperature, T_v
k	= thermal conductivity of mixture
$l(x_1,x_2)$	= characteristic dimension of roughness element
$\tilde{m}_D$	= nondimensional drag parameter, Eq. (18)
$n(x_1)$	= number of roughness sites per unit area
p	= pressure
r	= local recession rate
T	= temperature
T_v	= vaporization temperature
u_k	= velocity component in kth coordinate direction
v_s	= overall recession rate of solid surface
$\dot{w}$	= rate of production of sublimed vapor per unit volume
W_i	= molecular weight of species i
x_1,x_2,x_3	= Cartesian coordinates, Fig. 1
Y_i	= mass fraction of species i
α	= fraction of cross-sectional area occupied by solid
Γ	= roughness parameter, Eq. (15)
ϵ	= expansion parameter
η	= similarity variable, Eq. (10)
η_r	= height of the roughness elements in terms of η
μ	= viscosity of mixture
ρ	= mass density

Subscripts

e	= external stream
g	= gas
s	= solid
so	= conditions in the solid at x_2, $\eta \sim -\infty$
$1,2$	= external gas and sublimed vapor, respectively

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*Consultant. Also Professor of Fluid Mechanics, University of California, San Diego. Fellow AIAA.

Introduction

THERE is an extensive literature on the interaction of smooth, subliming surfaces under conditions of aerodynamic heating.[1,2] Here we consider a situation wherein the surface has a structure such that as it sublimes, a rough surface develops; thus the interaction between the solid and the gas stream is more complex. We assume that the structure of the solid leads to a specified number of roughness sites, denoted $n(x_1)$, where x_1 is the Cartesian coordinate along the surface as indicated in Fig. 1. The origin of the normal coordinate x_2 is placed at the bottom of the roughness elements. The shape and height of the resultant roughness elements are assumed to be described in a statistical sense by a function $\alpha(x_1,x_2)$ which gives the percent of total area in a plane normal to the x_2 axis occupied by solid. Accordingly, $\alpha(x_1,x_2 \leq 0)=1$, $\alpha(x_1,\infty)=0$. The distribution of $\alpha(x_1,x_2)$ is to be calculated.

In order to simplify the analysis without discarding essentials, we assume that the material sublimes and yields a vapor which does not react with the fluid in the external flow and, thus, that the structure of the subliming surface, resulting in the development of roughness, leads only to a simple product of decomposition. Accordingly, we deal with a tertiary system: external gas, vapor, and solid. Clearly, in this initial treatment of the interaction we ignore the aerothermochemical complexity which might be associated with the detailed structure of the subliming surface causing the roughness sites.

There are several approaches to the treatment of mixtures of gases and solids. In Williams[3] the fluid density ρ_f is the gas density ρ_g divided by the total volume, gas plus solid, and appropriate conservation equations from this point of view are given. In Kuo et al.,[4,5] the burning of porous propellants is studied with ρ_g and the porosity, $1-\alpha$ in our notation, used in the conservation equations. This latter approach is used by Chen et al.,[6] and we adopt it here. Thus, the partial densities

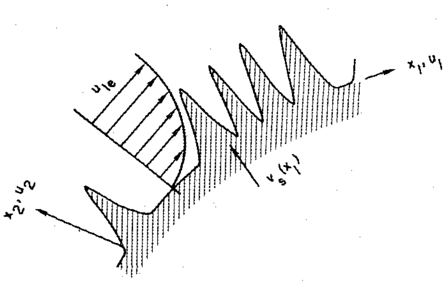


Fig. 1 Schematic representation of the flow.

of the gases, ρ_1, ρ_2, refer to the gas volume alone, $\rho_g = \rho_1 + \rho_2$, and the mass fractions $Y_i = \rho_i / \rho_g$; $i = 1, 2$; $Y_1 + Y_2 = 1$. The quantity ρ_s is the mass density of the solid based on the solid volume alone.

As a further preliminary we remark that a length characterizing in a statistical sense the lateral dimension of the roughness elements can be defined in terms of α. If we assume that the elements are circular in cross section, we would have $l(x_1, x_2) = (4\alpha/\pi n)^{1/2}$; if the cross section of the element is square, $l(x_1, x_2) = (\alpha/n)^{1/2}$. For present purposes, the difference resulting from these two assumptions is inessential; accordingly, we employ the latter. Finally, we note that we treat only laminar flow and do not consider the effect of roughness on either transition or on the turbulent boundary layer.

Conservation Equations

We follow closely the development of Kuo et al.[4,5] to establish the appropriate conservation equations, making minor changes in order to simplify the analysis and to specialize the resulting equations according to the boundary-layer approximation. Consider first the conservation of species; we find

$$\frac{\partial}{\partial x_k}\left[\rho_g Y_i u_k (1-\alpha)\right] = \frac{\partial}{\partial x_k}\left[\rho_g (1-\alpha)\mathfrak{D}\frac{\partial Y_i}{\partial x_k}\right] + \dot{w}\delta_{i2}$$

$$i = 1,2 \tag{1}$$

where the subscript 1 denotes the gas in the external stream and 2 denotes the vapor; $\dot{w}$ is the mass of vapor added per unit volume and time; and δ_{ij} is the Kronecker delta. The velocity component u_k is best considered as the velocity which, multiplied by the gas density and the area of the voids per unit total cross-sectional area normal to the kth coordinate, equals the mass flux. When the mass flux and the cross-sectional area of the voids both vanish, the velocity component is undefined.

The source term is described as in Kuo et al.,[4,5] i.e., according to the approximation

$$\dot{w} = 4r\ell n = 4r(\alpha n)^{1/2} \tag{2}$$

where r is the surface recession rate, e.g., in gm/cm^2 s.

Relative to an x_2 coordinate fixed on the bottom of the roughness elements, the solid steadily advances with a velocity $v_s(x_1)$. Accordingly, conservation of solid is given by the equation

$$\frac{\partial}{\partial x_2}(\rho_s v_s \alpha) = -\dot{w} = \rho_s v_s \frac{\partial \alpha}{\partial x_2} \tag{3}$$

Overall continuity is obtained by adding Eq. (1) with $i = 1, 2$ to Eq. (3) and by noting $Y_1 + Y_2 = 1$; we find

$$\frac{\partial}{\partial x_k}\left[\rho_g (1-\alpha) u_k\right] + \rho_s v_s \frac{\partial \alpha}{\partial x_2} = 0 \tag{4}$$

Note that, if $\alpha \equiv 0, 1$, Eqs. (1) and (4) degenerate to the usual species conservation and continuity equations and to identities, respectively. Equation (1) with $i = 1$ is preferred because the source term is absent; we can always use $Y_2(x_1, x_2) = 1 - Y_1(x_1, x_2)$ to obtain the mass fraction of vapor. In the boundary-layer approximation, we have for Eq. (1) with $i = 1$

$$\frac{\partial}{\partial x_k}\left[\rho_g Y_1 u_k (1-\alpha)\right] = \frac{\partial}{\partial x_2}\left[\rho_g (1-\alpha)\mathfrak{D}\frac{\partial Y_1}{\partial x_2}\right] \tag{5}$$

Equations (2-5) represent mass conservation for the present flow.

We consider momentum conservation only in the x_1 direction. However, in doing so we must include the effect of the drag of the roughness elements. We follow Kuo et al.[4,5] and introduce the drag per unit wetted area of roughness elements, denoted D. The injected vapor is assumed to contribute no increment of momentum. Thus we have

$$\frac{\partial}{\partial x_k}\left[\rho_g (1-\alpha) u_k u_1\right] = -(1-\alpha)\frac{\partial p}{\partial x_1}$$

$$+ \frac{\partial}{\partial x_2}\left[\mu (1-\alpha)\frac{\partial u_1}{\partial x_2}\right] - 4(\alpha n)^{1/2} D \tag{6}$$

If $\alpha = 0$, again we recover the well-known boundary-layer equation. For $\alpha = 1$ and $D = 0$, the equation degenerates to an identity.

Because of the use of gas density based on gas volume, not on total volume, the equation of state is

$$p = \rho_g R_0 T_g \left(\frac{Y_1}{W_1} + \frac{Y_2}{W_2}\right) = \frac{\rho_g R_0 T_g}{W_1}\left[Y_1(1-w) + w\right] \tag{7}$$

where $w \equiv W_1 / W_2$.

We consider next an overall energy balance. If we assume the Prandtl and Lewis numbers for the gas are unity, and consider a low-speed flow, we obtain

$$\frac{\partial}{\partial x_k}\left[\rho_g (1-\alpha) u_k h\right] + \rho_s v_s c_s \frac{\partial}{\partial x_2}(\alpha T_s)$$

$$= \frac{\partial}{\partial x_2}\left[\mu (1-\alpha)\frac{\partial h}{\partial x_2}\right] + k_s \frac{\partial}{\partial x_2}\left(\alpha \frac{\partial T_s}{\partial x_2}\right) \tag{8}$$

where c_s and k_s are the specific heat capacity and thermal conductivity of the solid, respectively; $T_s(x_1, x_2)$ is an average temperature of the solid; and h is the enthalpy of the gas per unit mass of gas. For $\alpha = 0$, we obtain the usual energy equation for a low-speed laminar boundary layer while for $\alpha \equiv 1$ we obtain the energy balance in a solid which is moving with a velocity v_s.

We need as a final conservation equation the energy balance in a roughness element. The flux to the surface involves a loss due to vaporization and the gain due to convective heating. We describe the latter in terms of a nondimensional heat-transfer coefficient h_c, assumed given. To simplify the analysis without loss of essentials we make the heat loss due to vaporization equal to the mass loss per unit volume times the enthalpy of the vapor at an assumed vaporization temperature T_v. This is closely related to the assumption of Kuo et al.[4,5] who uses for the enthalpy of the vapor the "enthalpy of gas at adiabatic flame temperature," i.e., a constant. Thus, we have

$$\rho_s v_s c_s \frac{\partial}{\partial x_2}(\alpha T_s) = k_s \frac{\partial}{\partial x_2}\left(\alpha \frac{\partial T_s}{\partial x_2}\right) - \dot{w} h_v + \mu_e h_c n (h - h_v) \tag{9}$$

For $\alpha = 1$, this equation becomes the equation for heat conduction in a moving, one-dimensional solid with heat loss due to vaporization and heat gain due to convection.

Equations (3-9) supplemented by expressions for r, D, and h_c; by a description of the pressure distribution, $p(x_1)$; and by physical and thermochemical data for n, ρ_s, W_1, W_2, c_s, μ, $\mathfrak{D}$, k_s, $h_1(T_g)$, $h_2(T_g)$, and h_v determine the following dependent variables as functions of x_1, x_2: ρ_g, Y_1, u_1, u_2, α, h, and T_s. The treatment of the roughness in terms of an average behavior appears to prevent explicit determination of the overall recession rate $v_s(x_1)$. Accordingly, we shall assume throughout that this rate is specified.

Similarity Equations

To make further progress we introduce the similarity variable $\eta = \eta(x_2)$ appropriate for the two-dimensional stagnation point† and a stream function to facilitate description of the velocities within the boundary layer. In this case, ρ_g, Y_1, u_2, α, h, and T_s depend only on x_2, and u_1 is an odd function of x_1.

Modified Stream Function

Consider the similarity variable and modified stream function according to the following definitions:

$$\eta = \left(\frac{\rho_e a}{\mu_e}\right)^{1/2} \int_0^{x_2} \frac{\rho_g}{\rho_e} dx_2' \tag{10}$$

$$(1-\alpha)\tilde{u}_1 = f'(\eta) \tag{11}$$

where $\tilde{u}_1 = u_1/ax_1 = u_1/u_e$.

If Eqs. (3) and (11) are introduced into Eq. (4) and variables changed, an integration with respect to η results in

$$\tilde{\rho}(1-\alpha)\tilde{u}_2 = -[f + \rho_s v_s/(\rho_e \mu_e a)^{1/2}\alpha] \tag{12}$$

where

$$\tilde{\rho} = \rho_g/\rho_e, \quad \tilde{u}_2 = (\rho_e a/\mu_e)^{1/2} u_2/a$$

are a nondimensional density and normal velocity, respectively. In Eq. (12), the constant of integration is selected so that

$$f(0) = -\rho_s v_s/(\rho_e \mu_e a)^{1/2} \tag{13}$$

The value, $f(0)$, at present, is considered to be specified and related to the gross sublimation rate, $\rho_s v_s$, via Eq. (13).

Roughness Elements

We consider first the distribution of the roughness elements in terms of the similarity variable, although η is an unnatural coordinate to describe the behavior of the solid. Consistent with Kuo et al.[4,5] and other analyses of two-phase flows, we make the simplifying assumption that the local recession rate r is a known constant.‡ Then Eqs. (2) and (3) lead to an explicit solution for $\alpha(x_2)$, namely to,

$$\alpha = \left[1 - 2\frac{r}{(\rho_e \mu_e a)^{1/2}} \frac{(\rho_e \mu_e a)^{1/2}}{\rho_s v_s} \left(\frac{\mu_e n}{\rho_e a}\right)^{1/2} \left(\frac{\rho_e a}{\mu_e}\right)^{1/2} x_2\right]^2 \tag{14}$$

or, in terms of η and in a form convenient for subsequent treatment,

$$\alpha = [1 - R(\eta)]^2 \tag{15}$$

where

$$R(\eta) = \Gamma \int_0^{\eta} \tilde{\rho}^{-1} dx_2$$

and

$$\Gamma = 2\frac{r}{(\rho_e \mu_e a)^{1/2}} \left(\frac{\mu_e n}{\rho_e a}\right)^{1/2} \frac{1}{[-f(0)]} = 2\left(\frac{\mu_e n}{\rho_e a}\right)^{1/2} \frac{r}{\rho_s v_s}$$

†The extension to the axisymmetric stagnation point is straightforward.

‡There is no difficulty encountered in extending the analysis to the case of a variable recession rate expressed as a function of either α or x_2. It would be via a functional form $r = r(\alpha)$ that a particular roughness shape and more of the details of the structure of the surface could be incorporated into the analysis. For example, a two-step form $r = r_0$ for $\alpha > \alpha_0$ and $r = r_1$ for $\alpha < \alpha_0$ might represent an impregnated fiber surface. In the present first treatment we assume that r is a representative average value which can be estimated either theoretically or from experiment.

In developing Eq. (15) we have used the inverse transformation corresponding to Eq. (10). Equation (14) indicates that for r, $f(0) \neq 0$ the roughness elements in this analysis are pyramidal in shape with a distribution of the characteristic length $l(x_2) = (1 - \Gamma(\rho_e a/\mu_e)^{1/2} x_2) n^{-1/2}$.

The parameter Γ plays a key role in our considerations; for example, note that the roughness height in terms of η is defined implicitly by the condition $R(\eta_r) = 1$ and thus that $\eta_r \propto \Gamma^{-1}$. Accordingly, if Γ is small in some sense, the roughness height is "large" while if Γ is large, the roughness height is small. Thus, we see that for a given overall recession rate, i.e., for a given $f(0)$, decreases in the local recession rate r and in the number of roughness elements n result in increases in the relative roughness height. These would appear to be intuitively correct implications. Note further that Γ involves the nondimensional quantities $r/(\rho_e \mu_e a)^{1/2}$ and $(\mu_e n/\rho_e a)^{-1}$ which represent, respectively, a recession rate parameter and a Reynolds number based on the number of roughness elements per unit area.

The definition of Γ can also be arranged as shown to involve the product of the roughness Reynolds number and the ratio of the local to bulk recession rates, $r/\rho_s v_s$. If we assume this latter factor to be constant for present purposes, the assumption that $\Gamma \sim \infty$ is interpreted to imply the number of roughness sites n increasing indefinitely. In this context it is not surprising that the limit $\Gamma \sim \infty$, $\eta_r \sim 0$, does not lead to a description of the laminar boundary layer on a smooth subliming surface. To obtain the well-known boundary-layer solutions for this latter case requires the a priori statement $\alpha(x_2) = 0$, $x_2 > 0$ and the abandonment of Eq. (14).

Finally, we note that, if we consider fixed roughness elements, i.e., those not due to differential sublimation but due simply to a rough surface, we consider that r is a function of η such that $R(\eta)$ in Eq. (15) describes the actual distribution of α. In this case the roughness elements need not be pyramidal.

A significant implication derives from Eq. (15); as $\eta \sim 0$ we have

$$1 - \alpha \sim 2\Gamma \tilde{\rho}^{-1}(0)\eta \tag{16}$$

Thus, from Eq. (11) we see that the usual boundary condition on $f'(0)$, namely, $f'(0) = 0$, implies that $\tilde{u}_1(0)$ need not be zero as suggested earlier. Subsequent consideration of the momentum equation in the neighborhood of the origin indicates that, as usual in boundary-layer flows, $f(\eta) \cong f(0) + f''(0)\eta^2/2 + 0(\eta^3)$ and, thus, that the average velocity at the origin is not zero, although the mass flux in the x_1 direction, being proportional to $(1-\alpha)\tilde{u}_1$ is indeed zero.

Species and Momentum Equation

We turn now to Eq. (5); in terms of similarity variables we have

$$[(1-\alpha)Y_1']' + [f - f(0)\alpha]Y_1' + [-f(0)\alpha']Y_1 = 0 \tag{17}$$

where we have for simplicity assumed $(\rho_g \mu/\rho_e \mu_e) = (\rho_g \mathfrak{D}/\mu) = 1$. Note that, if $\alpha \equiv 0$, we obtain the usual conservation equation.

Next, the streamwise momentum equation, Eq. (6), leads to

$$\begin{aligned}&\{(1-\alpha)[f'/(1-\alpha)]'\}' + [f - f(0)\alpha][f'/(1-\alpha)]' \\ &+ [(1-\alpha)/\tilde{\rho} - f'^2/(1-\alpha)] + [f'/(1-\alpha)] \\ &\times [-f(0)\alpha' - \tilde{m}_D \alpha/\tilde{\rho}^2] = 0\end{aligned} \tag{18}$$

If $\alpha \equiv 0$, we recover the usual momentum equation for the boundary layer at a two-dimensional stagnation point.

The drag term is the last one on the left side of Eq. (18) and requires some explanation. Kuo et al.[4,5] use, for the description of the drag in their study, information related to packed beds of spheres and include both viscous and profile drag effects. For the stagnation point only the former effect is operative. The quantity D thus becomes proportional to $\mu u_l(\alpha n)^{1/2}/(1-\alpha)$.[2] Strict application of this form would yield $\tilde{m}_D \propto (1-\alpha)^{-2}$, which leads to a physically unacceptable behavior at $\eta \sim 0$. The implication of this result is either that the packed bed data do not apply for $\alpha \sim 1$ or that the drag of cylinders in a dense array behaves differently from spheres in a packed bed. In the absence of experimental data to the contrary, we replace α in the derived form for $\tilde{m}_D$ with a mean value and obtain the indicated drag term with $\tilde{m}_D$ taken to be a specified constant. However,

$$\tilde{m}_D = m_D(\mu_e n/\rho_e a)$$

where m_D is estimated to be $O(10^2)$.

Aerothermochemistry

We consider next the aerothermochemical equations required to describe the system. For the stagnation point Eq. (7) may be written as

$$\tilde{\rho}\tilde{T}[w+(1-w)Y_l]=1 \tag{19}$$

where $\tilde{T}=T_g/T_e$. Consider next the enthalpy for the gas mixture; it is convenient to relate the heat of formation of the vapor to the external temperature T_e so that the heat of formation of the gas there is zero. Thus,

$$\tilde{h}\equiv h/h_e=\tilde{T}[w+(1-w)Y_l]+(\Delta_2/c_{pe}T_e)(1-Y_l) \tag{20}$$

where for simplicity we take the molar coefficient of specific heat $c_{pi}W_i$ to be constant. Equations (19) and (20) allow the density ratio $\tilde{\rho}(\eta)$ to be expressed as

$$\tilde{\rho}^{-1}=\tilde{h}-(\Delta_2/c_{pe}T_e)(1-Y_l) \tag{21}$$

The parameter involving Δ_2 relates the thermodynamics of the vapor given off by the solid to the enthalpy of the external stream.

Energy Equations

Consider next Eq. (8); we find in similarity variables

$$[(1-\alpha)\tilde{h}']'+[f-f(0)\alpha]\tilde{h}'+[-f(0)\alpha']\tilde{h}$$
$$-[-f(0)](c_sT_{so}/h_e)(\alpha\tilde{T}_s)'$$
$$+(k_s/\mu_e c_s)(c_sT_{so}/h_e)(\tilde{\rho}\alpha\tilde{T}_s')'=0 \tag{22}$$

where $\tilde{T}_s\equiv T_s/T_{so}$, $T_{so}\equiv T_s(-\infty)$ and where (c_sT_{so}/h_e) and $(k_s/\mu_e c_s)$ are nondimensional parameters relating the thermodynamic properties of solid and gas. Note that, if $\alpha=0$, we recover the usual energy equation applicable to stagnation point flow.

Finally, Eq. (9), determining the temperature distribution in the solid, becomes

$$(k_s/c_s\mu_e)(\tilde{\rho}\alpha\tilde{T}_s')'-[-f(0)](\alpha\tilde{T}_s)'$$
$$+(h_e/c_sT_{so})[-f(0)\alpha']\tilde{h}_v=0 \tag{23}$$

where $\tilde{h}_v=h_v/h_e$. In deriving Eq. (23) from Eq. (9) it is found that the usual form for the convective heat transfer in beds, e.g., the one employed by Kuo et al.,[4,5] leads to a term proportional to $x_l^{0.7}$. This must be compared to the other terms in the equation developed from Eq. (9), namely those independent of x_l, and must therefore be considered negligible. The implication of this result is that vaporization and internal conduction dominate the energy balance within roughness elements at a stagnation point.

An improved form of Eq. (22) results if it is combined with Eq. (23) so that

$$[(1-\alpha)\tilde{h}']'+[f-f(0)\alpha]\tilde{h}'+(\tilde{h}-\tilde{h}_v)[-f(0)\alpha']=0 \tag{24}$$

If $\alpha=0$, we recover the usual energy equation for stagnation point flows.

Mathematical Formulation and Boundary Conditions

Consideration of the analysis at this stage of presentation indicates that the calculation of the velocity, species, and energy distributions are independent of the temperature distribution in the roughness elements. Thus Eqs. (15, 17, 18, and 24) can be solved for $\alpha(\eta)$, $Y_l(\eta)$, $f(\eta)$, and $\tilde{h}(\eta)$. Subsequently, Eq. (23) may be solved for the temperature distribution within the solid. For the first stage in the calculation, a solution is identified with the overall recession rate by $f(0)$, with the two thermochemical parameters $\tilde{h}_v$ and $(\Delta_2/c_{pe}T_e)$, with the drag parameter $\tilde{m}_D$, and with the roughness parameter Γ.

The appropriate point of view regarding the describing equations for the first stage of the calculation is the following: two sets of equations are involved—those appropriate for $0\le\alpha\le 1$, $0\le\eta\le\eta_r$ and describing the flow involving roughness, and those for $\alpha\equiv 0$, $\eta>\eta_r$ and describing the boundary layer external to the roughness. The boundary conditions for the second set are clearly

$$f'(\infty)=\tilde{h}(\infty)=Y_l(\infty)=1 \tag{25}$$

Moreover, with $\alpha(\eta)$ given by Eq. (15), we see that $\alpha'(\eta_r)=0$ so that the solutions to the two sets of equations are continuous at $\eta=\eta_r$ in terms of the functions and all derivatives appearing in the differential equations.

To guide the selection of the boundary conditions at $\eta=0$, we assume plausible series expansions for the dependent variables in the neighborhood of the origin, substitute these into the conservation equations, and consider the first, nontrivial terms. Accordingly, we assume

$$\alpha\cong 1-2R'(0)\eta+O(\eta^2)$$
$$f\cong f(0)+f''(0)\eta^2/2+O(\eta^3)$$
$$\tilde{h}\cong\tilde{h}(0)+\tilde{h}'(0)\eta+O(\eta^2)$$
$$Y_l\cong Y_l(0)+Y_l'(0)\eta+O(\eta^2)$$

When substituted into Eqs. (17) and (24), the lowest terms in η yield the familiar boundary conditions

$$Y_l'(0)=-f(0)Y_l(0) \tag{26}$$

$$\tilde{h}'(0)=-f(0)[\tilde{h}(0)-\tilde{h}_v] \tag{27}$$

A similar substitution into Eq. (18) imples that $f''(0)$ is arbitrary so that the boundary conditions on $f(\eta)$ are $f(0)$ given and

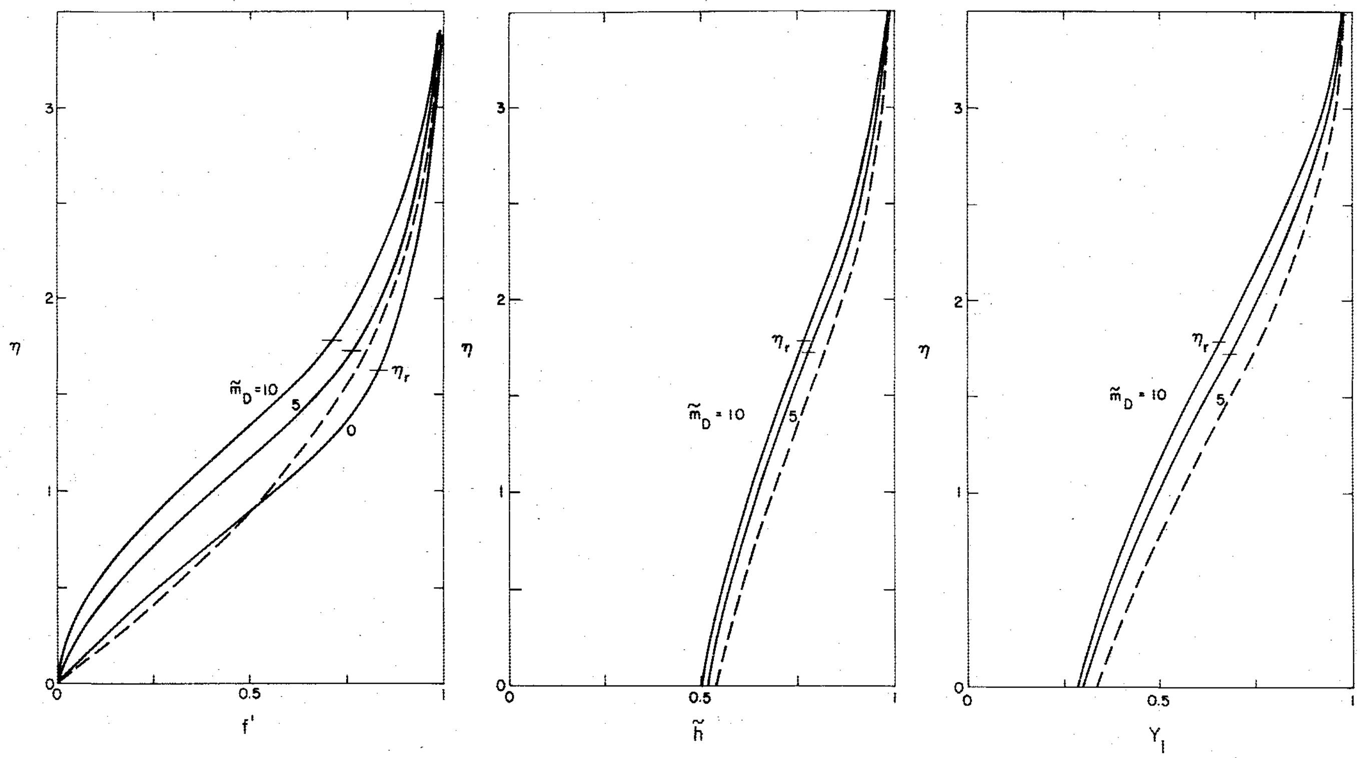
$$f'(0)=0 \tag{28}$$

Finally, the value of η_r, to be determined as part of the solution, is implicitly given by the condition

$$R(\eta_r)=1 \tag{29}$$

Temperature Distribution within the Roughness Elements

To complete the analysis, the temperature distribution within the roughness elements can be determined by integration of Eq. (23). However, in the interests of brevity we do not present the results of this calculation.



a) Velocity profiles b) Enthalpy profiles c) Concentration of gas in the external flow

Fig. 2 The profiles for various values of the drag parameter: $\Gamma=1, f(0)=-0.5$.———smooth surface.

Special Cases

Before discussing the complete solution to the equation given here, it is illuminating to consider special cases.

Large Number of Roughness Sites and Reduction to a Smooth Surface

If the Reynolds number $(\rho_e a/\mu_e n)$ is small, i.e., the number of roughness sites is large, the roughness parameter Γ is large and the extent of the roughness elements in terms of η is small. A perturbation analysis in terms of the parameter $\epsilon=\Gamma^{-1}$ is suggested; if, in the usual fashion, we introduce inner variables $\tilde\eta=\eta/\epsilon$, $\tilde f=[f-f(0)]/\epsilon$, etc., and if for expendiency we drop the drag term, the solutions for the inner layer are found to be

$$F(\tilde\eta)=F(0) \tag{30}$$

$$\tilde h(\tilde\eta)=h(0) \tag{31}$$

$$Y_l(\tilde\eta)=Y_l(0) \tag{32}$$

where $F(\eta)=f'/(1-\alpha)$. The implication from these solutions is that the mass averaged velocity, the enthalpy, and the species concentration within the inner layer are constant and that the inner layer is dominated by the change of α, i.e., by $\alpha'(\eta)$. The constants in Eqs. (30-32) are determined by matching with the outer solutions which correspond to solutions to the usual boundary-layer equations but with the following boundary conditions:

$$f''(0)+f(0)f'(0)=0 \tag{33}$$

$$\tilde h'(0)+f(0)[\tilde h(0)-\tilde h_v]=0 \tag{34}$$

$$Y_0'(0)+f(0)Y_l(0)=0 \tag{35}$$

The last two are the usual conditions applicable to a subliming surface, but the first corresponds to a slip condition as discussed earlier. The thickness of the inner layer is determined by substitution of the inner solutions into Eq. (29), namely by $\eta_r=\epsilon\tilde\eta_r=\epsilon\{\tilde h(0)-\Delta_2[1-Y_l(0)]\}^{-1}$.§

Small Number of Roughness Sites

The opposite limiting case arises when the Reynolds number $(\rho_e a/\mu_e n)\gg 1$, i.e., there are few roughness sites. In this case the extent of the roughness elements in terms of η is large, and we expect the boundary layer to have a thick inner layer and a relatively thin outer layer providing the adjustment to the external flow. This situation is reminiscent of the boundary layer with large rates of injection.[7,8] The exact numerical solutions suggest that, in fact, the outer layer is the uniform external flow and that the roughness elements extend to the "edge" of the boundary layer, a physically appealing picture.

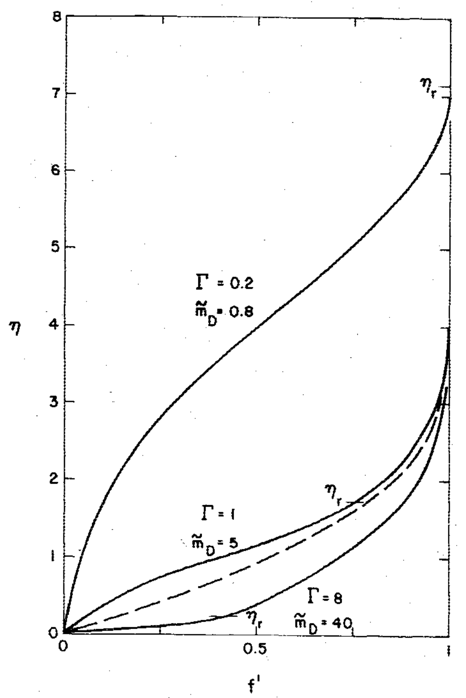
Fixed Roughness Elements

If the surface of the solid is nonsubliming but involves $n(x_l)$ roughness elements per unit surface area, then $f(0)=0$ and the function $R(\eta)$ must, in general, be altered to reflect the actual distribution of $\alpha(x_2)$. For consistency and simplicity, we assume that in this case the roughness elements of interest are, in fact, pyramidal so that the indeterminant quotient $[r/f(0)]$ is a constant, independent of η, and Γ is a specified but arbitrary constant. The boundary condition regarding $h(\eta)$ at $\eta=0$ must, in general, be altered from that given by Eq. (27) to reflect a different cooling mechanism in the absence of sublimation; for simplicity we shall assume here, that Eq. (27) continues to apply so that the solution of Eq. (24) is $\tilde h\equiv 1$. The solution for $Y_l(\eta)$ is clearly $Y_l(\eta)\equiv 1$, since there is no vapor introduced into the boundary layer.

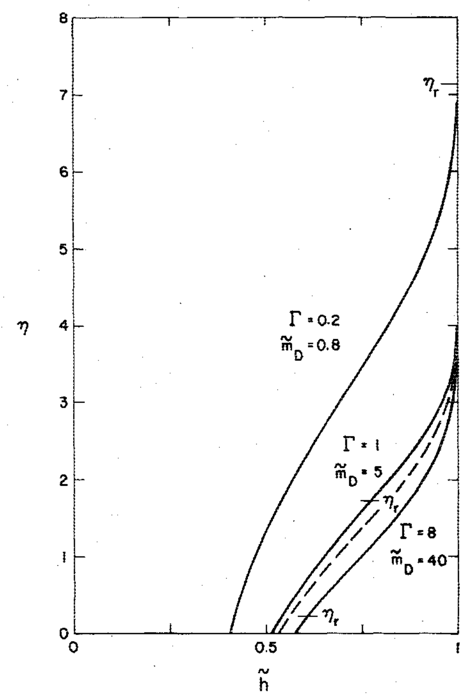
Numerical Results

The numerical analysis of the various sets of describing equations involved in this analysis is characterized by two- or

§The rigorous treatment of the perturbation problem outlined here involves some subtle, unconventional considerations. David R. Kassoy and the present author will publish the details of the full calculation elsewhere.



a) Velocity profiles



b) Enthalpy profiles

Fig. 3 The profiles for various values of the roughness parameter Γ with $m_D = 10^2$: $-f(0) = 0.5$.———smooth surface.

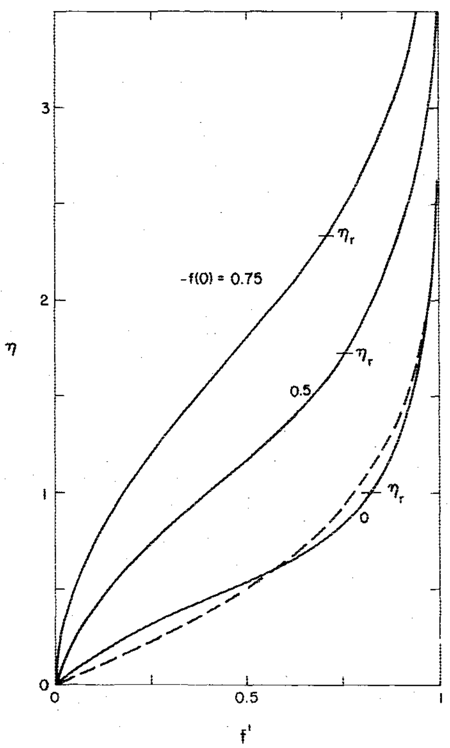


Fig. 4 The velocity profiles for various values of the overall recession rate, Γ = 1, $\tilde{m}_D = 5$.———smooth surface.

three-point boundary conditions applicable to a system of nonlinear ordinary differential equations. In the interest of brevity we do not give details of the numerical analysis but simply note that the method of quasilinearization as described elsewhere is employed and generally proves effective; difficulties in convergence similar to those experienced previously when the rates of injection become large are encountered when Γ becomes small.[7,8]

Boundary-Layer Characteristics

There are a large number of parameters involved in the solutions. Accordingly, we can only illustrate the nature of the effects by presenting results for representative values of these parameters. As to the aerothermochemical data, we take for all the results $\tilde{h}_v = 0.3$ and $(\Delta_2/c_{pe}T_e) = 0.1$, values which appear to relate to re-entry conditions for a polymetric ablator. Several values for the parameters determining the overall recession rate, the drag, and of the roughness are assumed to indicate their effect.

In Figs. 2a-c we show the effect of the drag parameter $\tilde{m}_D$ for Γ = 1, $-f(0) = 0.5$. In terms of the definitions of Γ and $\tilde{m}_D$, these results can be considered to show the effect of varying the ratio of the local to the overall recession rates, i.e., of varying $(r/\rho_s v_s)$. Also shown for comparison are the solutions for a smooth subliming surface. These figures indicate that, as might be expected, the drag of the roughness elements results in velocity profiles which are more inflected than those on a smooth surface. The effect of the altered velocity profiles on the profiles of energy and species concentration is not dramatic. It is possible to deduce from Fig. 2b and Eq. (27) that, as the roughness Reynolds number decreases so that the drag parameter $\tilde{m}_D$ increases while the ratio of the two recession rates decreases so that Γ remains constant, the convective heating to the virgin material decreases.

In Figs. 3a and b we show the effect of varying the roughness parameter with $-f(0) = 0.5$. For each value of Γ we have varied $\tilde{m}_D$ so that the ratio $(r/\rho_s v_s)$ is approximately the same for each case.¶ We let $m_D = 10^2$ for this purpose. The most significant result shown here is the extent of the roughness layer with Γ large and small. In the latter case, note that the flow beyond the roughness elements is essentially the external flow. For the more extended roughness elements, the values of the enthalpy and also, although not shown, the concentration of the gas in the external flow near the bottom of the roughness elements, decrease. Note that in this case the convective heating to the virgin material as indicated by Eq. (27) increases as ratio of the two rates of recession increases; the implication from this result is that for a given number of

¶For Γ = 0.2 we should have $\tilde{m}_D = 1$, but our numerical analysis encounters difficulties as $\tilde{m}_D$ increases and Γ decreases, and we therefore settle for the results shown.

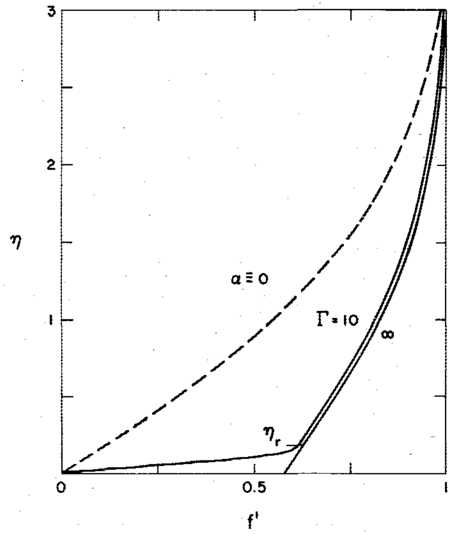


Fig. 5 Comparison of the velocity profiles for $\Gamma = 10$ with the outer layer solution corresponding to $\Gamma \sim \infty$, $-f(0) = 0.5$, $m_D = 0$. - - - smooth surface.

roughness sites the heating to the surface is reduced by making the local recession rate to the roughness elements small.

We show in Fig. 4 for $\Gamma = 1$, $\tilde{m}_D = 5$, the effect on the velocity profiles of varying the overall recession rate. Also shown for comparison is the velocity profile on a smooth, nonsubliming surface. For the nonsubliming case we retain Eq. (27) so that the flow corresponds to one of constant density. We see that the effect of increased overall recession rate augments that of the roughness elements so the profiles are highly inflected.

Finally, we show in Fig. 5 comparison of the solutions for $\Gamma = 10$, a value resulting in roughness elements of small extent in terms of η, namely $\eta_r = 0.179$, and the outer solution which corresponds to $\Gamma \sim \infty$ both with $m_D = 0$. There is good agreement between the two solutions. Note the significant influence of the roughness on the outer boundary layer even though the roughness elements are of negligible extent.

Concluding Remarks

We have carried out a calculation of the laminar boundary layer at a two-dimensional stagnation point of a surface which as it sublimes creates roughness and thus leads to a more complex interaction than usually considered. The behavior of the boundary layer is found to depend on a roughness parameter Γ which combines a Reynolds number based on the number of roughness elements per unit area and the ratio of the local to overall recession rates. For $\Gamma \ll 1$ the roughness elements determine the thickness of the boundary layer. For $\Gamma \sim \infty$ the roughness elements are of negligible extent but nevertheless lead to a stress which influences the boundary layer beyond the roughness elements.

Acknowledgment

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